THE RADIATION AND DIFFRACTION OF SURFACE WAVES FROM A VERTICALLY FLOATING PLATE

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The simplest plane problem connected with the hydrodynamics of wave motion of ships is dealt with in this article, namely the oscillation of a plate floating vertically in a disturbed heavy liquid of infinite depth. The problem is solved in "closed" form and the solution is obtained by the same methods as employed in the case of pure diffraction [1]. Exact values are obtained for general coefficients of damping and mass coupling, in terms of cylindrical functions. Formulas are derived giving mean values of the hydrodynamic forces over a cycle of oscillation in the form of a quadratic approximation.

1. Method of solution. We deal with a dense incompressible liquid of infinite depth on which there is a regular system of travelling waves determined by the following velocity potential;

$$\Phi_0(y, z, t) = -jcr_0 \exp[j\sigma t - \nu(z + jy)]$$

where $j = \sqrt{-1}$, σ is the frequency of oscillation, $2r_0$ is the wave height, $\nu = \sigma^2/g$, the wave number, g is the acceleration due to gravity, and $c = g/\sigma$, the phase velocity. Both here, and in the complex expressions to follow containing exp $j\sigma t$, only the real part will be considered.

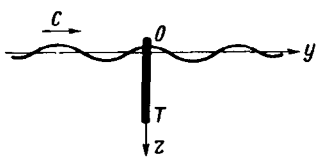

Fig. 1.

Now let us assume that the vertical floating plate (Fig. 1) performs small harmonic oscillations at frequency σ in the disturbed liquid with horizontal and angular velocities equal, respectively, to $V = v$ exp $j\sigma t$ and $\Omega = \omega$ exp $j\sigma t$. Also let $\Phi = (\phi + \phi_0)$ exp $j\sigma t$ be the velocity potential of the whole wave motion in the liquid, where $\phi_0 = jcr_0 \times \exp(-\nu(z + jy))$. Then we have the following boundary conditions for determining the harmonic function $\phi(y, z)$;

$$\frac{\partial\varphi}{\partial z} + \nu\varphi = 0 \quad \text{for} \quad z = 0 \tag{1.1}$$

$$\frac{\partial\varphi}{\partial y} = v + \omega z + \sigma r_0 e^{-\nu z} z \quad \text{for} \quad y = 0 \text{ and } 0 \leqslant z \leqslant T \tag{1.2}$$

To these we add the further radiation condition for the motion proceeding in both directions from each face of the plate;

$$\varphi(y, z) = jB_{\pm} e^{-\nu(z \pm jy)} \qquad \text{for } y \to \pm\infty \tag{1.3}$$

A further condition is that of the boundedness of the derivatives of ϕ within the region occupied by the fluid, and also their tending to zero for $z \to \infty$. $B_{\pm}$ represents complex quantities in j which have yet to be determined. Note also that the function ϕ can also be represented in a form suitable for computation, namely;

$$\varphi = v\varphi_2 + \omega\varphi_4 + \varphi_7, \quad \varphi_m = jB_m^{\pm} e^{-\nu(z \pm jy)} \qquad \text{for } y \to \pm\infty$$

$$\frac{\partial\varphi_2}{\partial y} = 1, \quad \frac{\partial\varphi_4}{\partial y} = z, \quad \frac{\partial\varphi_7}{\partial y} = \sigma r_0 e^{-\nu z} \quad \text{for } y = 0 \text{ and } 0 \leqslant z \leqslant T \tag{1.4}$$

$$\frac{\partial\varphi_m}{\partial z} + \nu\varphi_m = 0 \qquad \text{for } z = 0 \ (m = 2, 4, 7)$$

where ϕ_2 and ϕ_4 are functions of the radiation which represent the simplest forms of wave motion with plate oscillations of unit velocity amplitude, ϕ_1 is a dispersion function which gives a solution to the diffraction problem and $B_m^{\pm}$ $(m = 2, 4, 7)$ are asymptotic characteristics of the radiation and dispersion functions which determine the complex amplitudes of the radiated and the dispersed waves.

We now introduce the function $w = \phi + i\psi$ of the complex variable $x = z + iy$, where the operator $i = \sqrt{-1}$ is not interchangeable with operator j. Using this function, condition (1.1) takes the following form:

$$\mathrm{Re}\left(\frac{dw}{dx} + \nu w\right) = 0 \qquad \text{for } x = iy \tag{1.5}$$

on the basis of which we continue the function $dw/dx + \nu w$ into the upper half plane and from this we find that the given function is holomorphic and single valued outside the section $(-T, T)$ on the z-axis; at points symmetrical with respect to the y-axis $\mathrm{Re}(dw/dx + \nu w)$ takes on values numerically equal but opposite in sign, while $\mathrm{Im}(dw/dx + \nu w)$ retains the same values at these points. Moreover, in view of the fact that $\partial\phi/\partial y$

and ψ are continuous when going through the section $(0, T)$ on the z-axis, Im $(dw/dx + \nu w)$ is also continuous there. Thus

$$\int\limits_{C_0} \left(\frac{dw}{dx} + \nu w\right) dx = 0 \tag{1.6}$$

where C_0 is a contour which encloses the section $(-T, T)$. Bearing in mind equation (1.6) we see that in the neighbourhood of the point at infinity the following expansion is valid:

$$\frac{dw}{dx} + \nu w = \frac{ic_1}{x^2} + \frac{c_2}{x^3} + \ldots \tag{1.7}$$

where c_1, c_2, ... are real constants with respect to i.

Now let us take a look at another function $f(x) = r + is$ which is related to $w(x)$ through the differential equation

$$f = \frac{dw}{dx} + \nu w \tag{1.8}$$

From (1.8) we obtain

$$-s = \frac{\partial \varphi}{\partial y} - \nu \psi$$

and because at the section $(0, T)$

$$\frac{\partial \varphi}{\partial y} = -\frac{\partial \psi}{\partial z} = v + \omega z + \sigma r_0 e^{-\nu z}$$

$$\psi = \psi_1 - v(z - T) - \frac{\omega}{2}(z^2 - T^2) + \frac{\sigma r_0}{\nu}(e^{-\nu z} - e^{-\nu T})$$

where ψ_1 is the value of ψ at the point $z = T$, for s on the section $(0, T)$ we have the condition

$$-s = A + Bz + Cz^2 \tag{1.9}$$

$$A = \sigma r_0 e^{-\nu T} - \nu \psi_1 + v(1 - \nu T) - \tfrac{1}{2}\nu T^2 \omega, \qquad B = \omega + \nu v, \qquad C = \tfrac{1}{2}\nu\omega \tag{1.10}$$

In accordance with analytical extrapolation, from condition (1.5) the value of s at $(0, -T)$ should be made equal to the corresponding values of s at $(0, T)$. The function $f(x)$, which satisfies equations (1.6) and (1.9), is determined in the form [2]:

$$f(x) = -\frac{1}{\pi i \sqrt{x^2 - T^2}} \int\limits_{-T}^{T} \frac{s\sqrt{T^2 - z^2}}{z - x} dz$$

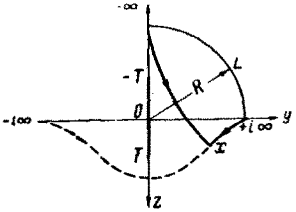

Fig. 2.

Taking into consideration the nature of function s on the section $(-T, T)$, and carrying out the calculation, we find

$$f(x) = iA\,\frac{x - \sqrt{x^2 - T^2}}{\sqrt{x^2 - T^2}} + i\,\frac{2BT}{\pi}\,\frac{x}{\sqrt{x^2 - T^2}}\left(1 - \frac{1}{T}\sqrt{x^2 - T^2}\,\mathrm{arc\,tg}\,\frac{T}{\sqrt{x^2 - T^2}}\right) +$$

$$+ iC\left(\frac{x^3 - {}^1/_2\,T^2 x}{\sqrt{x^2 - T^2}} - x^2\right) \tag{1.11}$$

From the differential equation (1.8) we find

$$w(x) = e^{-\nu x}\left(A_1 + iA_2 + \int\limits_{-\infty}^{x} f(x)\,e^{\nu x}\,dx\right) \tag{1.12}$$

where A_1 and A_2 are integration constants; the contour of integration $(-\infty, x)$ is shown in Fig. 2. It can be easily seen that

$$\lim_{R\to\infty}\int\limits_L f(x)\,e^{\nu x}\,dx = 0, \qquad \int\limits_{-\infty}^{x} f(x)\,e^{\nu x}\,dx = \int\limits_{i\infty}^{x} f(x)\,e^{\nu x}\,dx$$

where L is a quadrant of radius R (Fig. 2). If we make use of the last of these equations we get the following asymptotic relations

$$w(x) = (A_1 + iA_2)\,e^{-\nu x} \quad \text{as} \quad x \to i\infty, \qquad w(x) = (B_1 + iB_2)\,e^{-\nu x} \quad \text{as} \quad x \to -i\infty$$

where B_1 and B_2 are determined from the following relation

$$B_1 + iB_2 = A_1 + iA_2 + \int\limits_{i\infty}^{-i\infty} f(x)\,e^{\nu x}\,dx$$

If, in this formula, we replace the contour of integration by contour C_0 which encloses the section $(-T, T)$, and if we reduce that contour C_0 to the section we will have

$$B_1 + iB_2 = A_1 + iA_2 + 2\int\limits_{-T}^{T} r_+ e^{\nu z}\,dz \tag{1.13}$$

where r_+ is the value of the function r when approaching the section $(-T, T)$ from the side $y > 0$.

In order to satisfy condition (1.3) for the radiation of the outgoing wave system we should put

$$A_1 = jA_2 = jB_+, \qquad B_1 = -jB_2 = jB_- \tag{1.14}$$

Then, if in (1.13) we replace i by j and then by $-j$ we get

$$jB_{\pm} = \mp \int_{-T}^{T} r_+ e^{\nu z}\, dz \tag{1.15}$$

From relation (1.11) we get

$$r_+ = A\frac{z}{\sqrt{T^2 - z^2}} + \frac{2BT}{\pi}\left(\frac{z}{\sqrt{T^2-z^2}} - \frac{z}{2T}\ln\frac{T+\sqrt{T^2-z^2}}{T-\sqrt{T^2-z^2}}\right) + C\frac{z^3 - \frac{1}{2}T^2 z}{\sqrt{T^2-z^2}} \tag{1.16}$$

If we insert this expression into (1.15) and we make use of the integral representation of Bessel Functions of imaginary argument, we find

$$jB_{\pm} = \mp\left[AT\pi I_1(\mu) + \frac{2BT^2}{\pi}b_2 + CT^3 b_4\right] \qquad (\mu = \nu T) \tag{1.17}$$

$$b_2 = \pi\left[I_1(\mu) + \frac{1}{\mu^2}I_0^{-1}(\mu) - \frac{1}{\mu}I_0(\mu)\right]$$
$$b_4 = \pi\left[\left(\frac{1}{2} + \frac{2}{\mu^2}\right)I_1(\mu) - \frac{1}{\mu}I_0(\mu)\right] \tag{1.18}$$

In these expressions $I_n(\mu)$ is a Bessel Function of imaginary argument

$$I_n(\mu) = \frac{\mu^n}{\pi(2n-1)!!}\int_{-1}^{1}(1-u^2)^{n-1/2}e^{\mu u}\,du$$

$$I_0^{-1}(\mu) = \frac{1}{\pi}\int_{-1}^{1}\frac{\operatorname{sh}\mu u\,du}{u\sqrt{1-u^2}} = \int_0^{\mu} I_0(\mu)\,d\mu \tag{1.19}$$

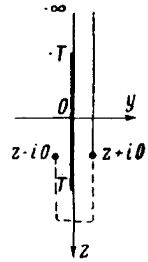

Fig. 3.

The expression for A contains an unknown constant ψ_1. To determine it we put $x = z$ in (1.12) and bear in mind the expressions (1.14) and (1.15); we then obtain

$$\varphi + i\psi = e^{-\nu z}\left[\pm\int_{T}^{z} r_+ e^{\nu z}\,dz + iB_+ + i\int_{-T}^{T} s e^{\nu z}\,dz + i\int_{T}^{z} s e^{\nu z}\,dz + \int_{-\infty}^{-T} f e^{\nu x}\,dx\right] \tag{1.20}$$

where the plus sign denotes approach of the section $(0, T)$ from the side $y > 0$ and the minus sign corresponds to $y < 0$. (Fig. 3). In (1.20) if we put $z = T$ and separate the imaginary part we have

$$e^{\mu}\psi_1 = B_+ + \int_{-T}^{T} s e^{\nu z}\,dz + \eta, \qquad \eta = -i\int_{-\infty}^{-T} f(x)e^{\nu x}\,dx \tag{1.21}$$

where η is a real constant. Taking into account the fact that for $z < -T$ the equation $(z^2 - T^2)^{1/2} = -(|z^2 - T^2|)^{1/2}$ holds and if we make use of the integral representation of the modified Hankel functions

$$K_n(\mu) = \frac{\mu^n}{(2n-1)!!} \int_1^\infty e^{-\mu u} (u^2 - 1)^{n - \frac{1}{2}} du$$

we find the following for η

$$\eta = AT\left(K_1(\mu) - \frac{e^{-\mu}}{\mu}\right) + \frac{2BT^2}{\pi}\left[K_1(\mu) - \frac{\pi}{2\mu^2}(1+\mu)e^{-\mu} + \right.$$
$$\left. + \frac{\pi}{2\mu^2} + \frac{1}{\mu}K_0(\mu) - \frac{1}{\mu^2}K_0^{-1}(\mu)\right] +$$
$$+ CT^3\left[\left(\frac{1}{2} + \frac{2}{\mu^2}\right)K_1(\mu) + \frac{1}{\mu}K_0(\mu) - \frac{e^{-\mu}}{\mu^3}(\mu^2 + 2\mu + 2)\right] \qquad (1.22)$$

$$\left(K_0^{-1}(\mu) = \int_0^\mu K_0(\mu)\, d\mu\right)$$

Thus, making use of (1.9), (1.10), (1.17), (1.21) and (1.22) we get the following expression for the constant ψ_1

$$\psi_1 = \frac{A_0 T}{\mu}\left(1 - \frac{e^{\mu}}{a}\right) + \frac{2BT^2}{\pi}\frac{b_1 + jb_2}{a} + CT^3\frac{b_3 + jb_4}{a} \qquad (1.23)$$

$$a = a_1 + ja_2 = \mu[K_1(\mu) + \pi j I_1(\mu)],$$
$$A_0 = \sigma r_0 e^{-\mu} + v(1-\mu) - \frac{1}{2}\mu T\omega \qquad (1.24)$$

$$b_1 = K_1(\mu) + \frac{1}{\mu}K_0(\mu) - \frac{1}{\mu^2}K_0^{-1}(\mu) - \frac{\pi}{2\mu^2}(1 + (\mu - 1)e^{\mu}) \qquad (1.25)$$

$$b_3 = \left(\frac{1}{2} + \frac{2}{\mu^2}\right)K_1(\mu) + \frac{1}{\mu}K_0(\mu) - \frac{e^{\mu}}{\mu^3}(\mu^2 - 2\mu + 2) \qquad (1.26)$$

The expressions (1.10) and (1.23) fully determine constant A, for which we have

$$A = \frac{1}{a}\left[A_0 e^{\mu} - \frac{2B\mu T}{\pi}(b_1 + jb_2) - C\mu T^2(b_3 + jb_4)\right] \qquad (1.27)$$

Now inserting the values of constants A, B and C from (1.10) and (1.24) to (1.27) into (1.17) and, bearing in mind the identity

$$I_0(\mu)K_1(\mu) + K_0(\mu)I_1(\mu) = \frac{1}{\mu} \qquad (1.28)$$

we find the final formulas for the asymptotic characteristics $B_m{}^{\pm}$ of the functions of radiation and dispersion:

$$B_2{}^{\pm} = \pm\frac{2TS_1}{\pi I_1(\mu) - jK_1(\mu)}, \qquad B_4{}^{\pm} = \pm\frac{2T^2(S_1 - {}^1\!/_4\,\pi)}{\mu[\pi I_1(\mu) - jK_1(\mu)]}$$
$$B_7{}^{\pm} = \pm\frac{\pi\sigma r_0 T I_1(\mu)}{\mu[\pi I_1(\mu) - jK_1(\mu)]} \qquad (1.29)$$

$$S_1 = \frac{\pi}{2\mu}\,[I_1(\mu) + L_1(\mu)], \qquad L_1(\mu) = \frac{2}{\pi}\,[I_0^{-1}(\mu)\,K_1(\mu) + K_0^{-1}(\mu)\,I_1(\mu) - 1] \tag{1.30}$$

We will demonstrate that $L_1(\mu)$ is a first order Struve Function of imaginary argument. Actually, it is easily established from (1.30) that

$$\frac{d^2L_1}{d\mu^2} + \frac{1}{\mu}\frac{dL_1}{d\mu} - \left(1 + \frac{1}{\mu^2}\right)L_1 = \frac{2}{\pi}, \qquad L_1(0) = 0, \qquad \left(\frac{dL_1}{d\mu}\right)_{\mu=0} = 0$$

The Struve Function, with the initial conditions indicated, satisfies this same equation, and has the following integral representation [3]

$$L_1(\mu) = \frac{2}{\pi}\mu\int_0^1 \sqrt{1-u^2}\,\mathrm{sh}\,\mu u\,du \tag{1.31}$$

Thus the expression for S_1 can also be represented in the form

$$S_1 = \int_0^1 \sqrt{1-u^2}\,e^{\mu u}\,du \tag{1.32}$$

Let us establish one more relation for the first order Struve function $L_0(\mu)$ determined with the help of expression [3]

$$L_0(\mu) = \frac{2}{\pi}\int_0^1 \frac{\mathrm{sh}\,\mu u\,du}{\sqrt{1-u^2}} \tag{1.33}$$

From (1.33) and (1.31) we have

$$L_0(\mu) = \frac{2}{\pi}\mu + \int_0^{\mu} L_1(\mu)\,d\mu$$

Inserting expression (1.30) and integrating by parts, we find

$$L_0(\mu) = -\frac{2}{\pi}\,[I_0^{-1}(\mu)\,K_0(\mu) - K_0^{-1}(\mu)\,I_0(\mu)] \tag{1.34}$$

2. Linear expressions for the hydrodynamic forces. We will now work out the total hydrodynamic forces which act on the plate using the solution we have obtained to the problem in linear wave theory. Let Y represent the resultant of the hydrodynamic forces and let M be the resultant moment about the origin, we then have the formulas

$$Y = \int_0^T (p_- - p_+)\,dz, \qquad M = \int_0^T z\,(p_- - p_+)\,dz \tag{2.1}$$

where p_- is the pressure on the plate acting at the side $y < 0$ and p_+ is that from the side $y > 0$. Pressure in the liquid is determined with the aid of the following linearized expression where p

$$p - p_0 = -\rho j\sigma(\varphi + \varphi_0)e^{j\sigma t} + \rho g z$$

where p_0 is the atmospheric pressure and ρ is the density of the liquid. Making use of these expressions and of (1.20) we find

$$p_- - p_+ = \rho j\sigma(\varphi_+ - \varphi_-)e^{j\sigma t} = 2\rho j\sigma e^{j\sigma t - \nu z}\int_T^z r_+ e^{\nu z}\,dz \tag{2.2}$$

Now putting (2.2) into (2.1) and integrating by parts, we then get

$$Y = -\frac{2}{\nu}\rho j\sigma e^{j\sigma t}\int_0^T r_+(e^{\nu z}-1)\,dz \quad M = -\frac{2}{\nu}\rho j\sigma e^{j\sigma t}\int_0^T r_+\left(\frac{e^{\nu z}-1}{\nu} - z\right)dz \tag{2.3}$$

From (1.16) and (2.3) we find the final expressions

$$\begin{aligned} Y &= -\frac{2T}{\nu}\rho j\sigma e^{j\sigma t}\left(\mu A S_1 + \frac{2}{\pi}BTY_B + CT^2Y_C\right) \\ M &= -\frac{2T^2}{\nu}\rho j\sigma e^{j\sigma t}\left(A\left(S_1 - \frac{\pi}{4}\right) + \frac{2}{\pi}BTM_B + CT^2M_C\right) \end{aligned} \tag{2.4}$$

The following quantities define Y_B, Y_C, M_B and M_C :

$$\begin{aligned} Y_B &= \mu S_1 - \frac{1}{\mu}S_0 + \frac{1}{\mu^2}S_0^{-1} + \frac{1}{2}, \qquad Y_C = \left(\frac{1}{2}\mu + \frac{2}{\mu}\right)S_1 - \frac{1}{\mu}S_0 + \frac{1}{3} \\ M_B &= S_1 - \frac{\pi}{6} - \frac{1}{\mu^2}S_0 + \frac{1}{\mu^3}S_0^{-1} + \frac{1}{2\mu} \\ M_C &= \left(\frac{1}{2} + \frac{2}{\mu^2}\right)S_1 - \frac{1}{\mu^2}S_0 + \frac{1}{3\mu} - \frac{\pi}{16} \end{aligned} \tag{2.5}$$

$$S_0 = \int_0^1 \frac{e^{\mu u}\,du}{\sqrt{1-u^2}} = \frac{\pi}{2}(I_0(\mu) + L_0(\mu)), \qquad S_0^{-1} = \int_0^1 \frac{e^{\mu u}-1}{\sqrt{1-u^2}}\,du = \int_0^\mu S_0(\mu)\,d\mu$$

Making use of the values of constants A, B and C determined from (1.10), (1.18) and (1.24) to (1.27), we can then put (2.4) in a form suitable for calculation:

$$Y = Y_g + Y_u, \qquad M = M_g + M_u \tag{2.6}$$

where Y_g and M_g are the disturbing force and its moment respectively caused by the diffraction of the travelling waves round the plate

$$Y_g = -2\rho g r_0 T\frac{S_1}{\pi I_1(\mu) - jK_1(\mu)}e^{j\sigma t}, \qquad M_g = -2\rho g r_0 T^2\frac{S_1 - 1/4\pi}{\mu[\pi I_1(\mu) - jK_1(\mu)]}e^{j\sigma t} \tag{2.7}$$

while Y_u and M_u represent the hydrodynamic force and its moment caused by radiation of waves in the heavy liquid due to horizontal and rotational oscillations of the plate:

$$Y_u = -\mu_{22}\frac{dV}{dt} - \lambda_{22}V - \mu_{24}\frac{d\Omega}{dt} - \lambda_{24}\Omega$$
$$M_u = -\mu_{42}\frac{dV}{dt} - \lambda_{42}V - \mu_{44}\frac{d\Omega}{dt} - \lambda_{44}\Omega \qquad (V = ve^{j\sigma t},\ \Omega = \omega e^{j\sigma t}) \quad (2.8)$$

In this expression λ_{nm} and μ_{nm} are generalized damping and mass coupling coefficients, depending on the dimensionless frequency parameter $\mu = \sigma^2 T/g$. We have the following expressions for these coefficients

$$\mu_{22} - \frac{j}{\sigma}\lambda_{22} = 2\rho T^2\left[\frac{S_1}{a}\left((1-\mu)e^{\mu} - \frac{2\mu^2}{\pi}(b_1 + jb_2)\right) + \frac{2}{\pi}Y_B\right] \qquad (2.9)$$

$$\mu_{24} - \frac{j}{\sigma}\lambda_{24} = 2\rho T^3\left[\frac{2}{\pi\mu}Y_B + \tfrac{1}{2}Y_C - \frac{\mu S_1}{a}\left(\tfrac{1}{2}e^{\mu} + \frac{2}{\pi}(b_1 + jb_2) + \frac{\mu}{2}(b_3 + jb_4)\right)\right] \qquad (2.10)$$

$$\mu_{42} - \frac{j}{\sigma}\lambda_{42} = 2\rho T^3\left[\frac{2}{\pi}M_B + \frac{S_1 - 1/4\pi}{\mu a}\left((1-\mu)e^{\mu} - \frac{2\mu^2}{\pi}(b_1 + jb_2)\right)\right] \qquad (2.11)$$

$$\mu_{44} - \frac{j}{\sigma}\lambda_{44} = 2\rho T^4\left[\frac{2}{\pi\mu}M_B + \tfrac{1}{2}M_C - \frac{S_1 - {}^1/_4\pi}{a}\left(\tfrac{1}{2}e^{\mu} + \right.\right.$$
$$\left.\left. + \frac{2}{\pi}(b_1 + jb_2) + \frac{\mu}{2}(b_3 + jb_4)\right)\right] \qquad (2.12)$$

Using expressions (1.28) and (1.30) we obtain the final formulas:

$$\lambda_{22} = 4\rho\sigma T^2\frac{S_1^2}{\pi^2 I_1^2(\mu) + K_1^2(\mu)}, \qquad \lambda_{44} = 4\rho\sigma T^4\frac{(S_1 - {}^1/_4\pi)^2}{\mu^2(\pi^2 I_1^2(\mu) + K_1^2(\mu))} \qquad (2.13)$$

$$\lambda_{24} = \lambda_{42} = 4\rho\sigma T^3\frac{S_1(S_1 - {}^1/_4\pi)}{\mu(\pi^2 I_1^2(\mu) + K_1^2(\mu))} \qquad (2.14)$$

$$\mu_{22} = \frac{4\rho T^2}{\pi}\left[\frac{1}{2} - \frac{1}{\mu}S_0 + \frac{1}{\mu^2}S_0^{-1} - \frac{S_1\Gamma}{\mu(\pi^2 I_1^2(\mu) + K_1^2(\mu))}\right] \qquad (2.15)$$

$$\mu_{24} = \frac{4\rho T^3}{\pi}\left[\frac{\pi}{12} + \frac{1}{2\mu} - \frac{1}{\mu^2}S_0 + \frac{1}{\mu^3}S_0^{-1} - \frac{S_1\Gamma - ({}^1/_4\pi)\Gamma_0'}{\mu^2(\pi^2 I_1^2(\mu) + K_1^2(\mu))}\right] \qquad (2.16)$$

$$\mu_{42} = \frac{4\rho T^3}{\pi}\left[\frac{\pi}{12} + \frac{1}{2\mu} - \frac{1}{\mu^2}S_0 + \frac{1}{\mu^3}S_0^{-1} - \frac{(S_1 - {}^1/_4\pi)\Gamma}{\mu^2(\pi^2 I_1^2(\mu) + K_1^2(\mu))}\right] \qquad (2.17)$$

$$\mu_{44} = \frac{4\rho T^4}{\pi}\left[\frac{1}{2\mu^2} - \frac{\pi}{12\mu} - \frac{\pi^2}{64} - \left(\frac{1}{\mu^3} + \frac{\pi}{4\mu^2}\right)S_0 + \frac{1}{\mu^4}S_0^{-1} - \right.$$
$$\left. - \frac{(S_1 - {}^1/_4\pi)(\mu^{-1}\Gamma - ({}^1/_4\pi)\mu\gamma_2)}{\mu^2(\pi^2 I_1^2(\mu) + K_1^2(\mu))}\right] \qquad (2.18)$$

In these expressions Γ_1 Γ_0 and γ_2 refer to the quantities

$$\Gamma = \gamma_1 - \mu\gamma_2 - \tfrac{1}{2}\pi K_1(\mu), \qquad \gamma_1 = \pi^2 I_0^{-1}(\mu) I_1(\mu) - K_0^{-1}(\mu) K_1(\mu) \qquad (2.19)$$
$$\gamma_2 = \pi^2 I_0(\mu) I_1(\mu) - K_0(\mu) K_1(\mu), \qquad \Gamma_0 = \mu^2 S_1\gamma_2 - \mu S_0(\pi^2 I_1^2(\mu) + K_1^2(\mu))$$

Expressions (2.16) and (2.17) differ in external appearance. Actually it is easy to demonstrate, using (1.28), (1.30), (1.34) and the last of (2.5), that $\Gamma_0 = \Gamma$. Thus, direct calculations for this particular problem correspond to the general law of tensor symmetry of the damping and mass coupling coefficients [4].

In accordance with general theory the generalized damping coefficients are expressed through the asymptotic characteristics of the $B_m^{\pm}$ radiation function in the following form [5] :

$$\lambda_{nm.} = \frac{1}{2}\rho\sigma \operatorname{Re}(B_n^+\overline{B}_m^+ + B_n^-\overline{B}_n^-) \tag{2.20}$$

which has been obtained from energy considerations, and lines on top of the letters denote transition to the conjugate complex value with respect to the imaginary j. If we put the expression for $B_m^{\pm}$ from (1.29) into (2.20) we get relations which are identical to (2.13) and (2.14).

Finally, it is well known that the disturbing forces and moments are fully determined by the radiation function for any arbitrarily chosen system of diffracting waves, and, for the case of travelling waves these forces and moments are expressed only through the asymptotic characteristics of the radiation functions [6]. For waves travelling in the direction of the y-axis, with the z-axis pointing vertically downwards, we have the formulas

$$Y_g = \rho g r_0 B_2^- e^{j\sigma t}, \qquad M_g = \rho g r_0 B_4^- e^{j\sigma t} \tag{2.21}$$

which, because of (1.29), coincide with (2.7).

The results obtained here, therefore, can be seen to be fully consistent with the three laws in general hydrodynamic wave theory applied to ships. The establishment of this correspondence is closely associated with the new relations (1.30) and (1.34) for the Struve Functions $L_0(\mu)$ and $L_1(\mu)$. Note also that it follows from the general theory [4,5] that $\mu_{nn}(0) > \mu_{nn}(\infty)$. In particular from formula (2.15) we have

$$\mu_{22} = \frac{\pi}{2}\rho T^2, \quad \mu_{22}(\infty) = \frac{2}{\pi}\rho T^2 \qquad \text{or} \qquad \mu_{22}(0)/\mu_{22}(\infty) = \frac{\pi^2}{4}$$

3. Mean values of the hydrodynamic forces in quadratic approximation. It has been shown [7] that a solution of a problem based on linear wave theory can be used for calculating mean values of non-linear characteristics in quadratic approximation over a period of oscillation, and, in particular, for calculating mean values of hydrodynamic forces and moments. In this case the pressure should be represented by the full expression

$$p - p_0 = -\rho \frac{\partial \Phi}{\partial t} - \frac{1}{2}\rho |\nabla \Phi|^2 + \rho g z \tag{3.1}$$

while in formulas (2.1) the lower limit of integration should be taken from the disturbed level of the liquid, i.e.

$$Y = \int_{\zeta_-}^{T} p_- dz - \int_{\zeta_+}^{T} p_+ dz, \qquad M = \int_{\zeta_-}^{T} z p_- dz - \int_{\zeta_+}^{T} z p_+ dz$$

where $\zeta_\pm$ represents the increase in height of the disturbed liquid when approaching the plate respectively from the sides $y > 0$ and $y > 0$. In the expression for M the lower limit of integration can be taken as zero because the corresponding integrals within the limits zero to $\zeta_\pm$ give terms of the third order; thus, retaining only terms of the second order, we have

$$Y = Y_1 + Y_2 \tag{3.2}$$

$$Y_1 = \int_0^T (p_- - p_+)\, dz, \quad Y_2 = \int_0^{\zeta_+} p_+ dz_- - \int_0^{\zeta_-} p_- dz, \quad M = \int_0^T z(p_- - p_+)\, dz$$

For subsequent calculation of mean values over a period of oscillation $\tau = 2\pi/\sigma$, we will make use of the rule

$$(uv)^* = \frac{1}{2}\operatorname{Re}(u\bar{v}) \qquad \left(a^* = \frac{1}{\tau}\int_t^{t+\tau} a(t)\, dt\right) \tag{3.3}$$

where u and v are functions of time by virtue of the exponential time multiplier exp. $j\sigma t$. If we make use of this rule and expression (3.1) we get

$$Y_1^* = \frac{1}{4}\rho \int_0^T (\nabla\Phi_+ \cdot \nabla\bar{\Phi}_+ - \nabla\Phi_- \nabla\bar{\Phi}_-)\, dz$$

$$M^* = \frac{1}{4}\rho \int_0^T z(\nabla\Phi_+ \cdot \nabla\bar{\Phi}_+ - \nabla\Phi_- \cdot \nabla\bar{\Phi}_-)\, dz$$

The velocity potential $\Phi(y, z, t)$ comprises the sum of the velocity potential of the approaching waves $\Phi_0 = -jcr_0 \exp[j\sigma t - \nu(z + jy)]$ and the velocity potential of the disturbed liquid motion $\phi(y, z) \exp j\sigma t$, while the values of the functions ϕ and $\partial\phi/\partial z$ on both sides of the section $(0, T)$ are identical in magnitude but opposite in sign, and the values of $\partial\phi/\partial y$ are the same.

Making use of this fact, we get

$$Y_1^* = \rho\sigma r_0 \operatorname{Im}\int_0^T \frac{\partial\varphi_+}{\partial z} e^{-\nu z}\,dz, \qquad M^* = \rho\sigma r_0 \operatorname{Im}\int_0^T z\frac{\partial\varphi_+}{\partial z} e^{-\nu z}\,dz \tag{3.4}$$

For calculating Y_2 it is sufficient to determine the pressure in the liquid by means of the linearized expressions

$$p - p_0 = -\rho\frac{\partial\Phi}{\partial t} + \rho g z \qquad \left(\zeta = \frac{1}{g}\left(\frac{\partial\Phi}{\partial t}\right)_{z=0}, \qquad \Phi = (\varphi + \varphi_0)\,e^{j\sigma t}\right)$$

and, within the limits of integration of Y_2 we can replace $\phi(0,\ z)$ and $\phi_1(0,\ z)$ by their values at point $z = 0$. Bearing this in mind, we obtain

$$Y_2^* = \rho\sigma r_0 \operatorname{Im} \varphi_+(0,\,0)$$

Thus for the average lateral force we have

$$Y^* = \rho\sigma\nu r_0 \operatorname{Im}\int_0^T \varphi_+ e^{-\nu z}dz$$

If we now insert the value of ϕ_+ from (1.20) and take into consideration (1.15), we finally end up with the simple expression

$$Y^* = \tfrac{1}{2}\rho\sigma r_0 \operatorname{Re} B_+ \tag{3.5}$$

This expression can be obtained by another method which is derived from the general formula for the average hydrodynamic forces acting on a ship, established in [7] using the theorem of change of momentum. This formula, applicable to the plane problem, has this form

$$Y^* = \tfrac{1}{2}\rho\left\{\Big|_{y=-\infty}^{y=\infty}\int_0^\infty\left[\left(\frac{\partial\Phi}{\partial z}\right)^2 - \left(\frac{\partial\Phi}{\partial y}\right)^2\right]dz - \frac{1}{g}\left(\frac{\partial\Phi}{\partial t}\right)_{z=0}^2\right\}^*$$

Because when $y \to \pm\infty$ $\Phi(y,\ z,\ t) = \Phi(y,\ 0,\ t)\exp(-\nu z)$, the preceding formula is considerably simplified:

$$Y^* = -\frac{\rho}{8\nu}\Big|_{y=-\infty}^{y=\infty}\left[\nu^2|\varphi + \varphi_0|^2 + \left|\frac{\partial\varphi}{\partial y} + \frac{\partial\varphi_0}{\partial y}\right|^2\right]_{z=0}$$

On the basis of an asymptotic correlation (1.3) for the function $\phi(y,\ 0)$, we find the following general expression

$$Y^* = \tfrac{1}{2}\rho\sigma r_0 \operatorname{Re} B_+ + \tfrac{1}{4}\rho\nu\left(|B_-|^2 - |B_+|^2\right) \tag{3.6}$$

In the problem under review, as is evident from (1.15), $B_- = -B_+$, and thus we obtain (3.5) from (3.6).

To work out M^* from formula (3.4) we will first of all use the equation $\partial\phi/\partial z = -\nu\phi + r$, and then insert the value of ϕ_+ from (1.20) into (3.4) and integrate by parts. After this we obtain

$$M^{*} = -\frac{1}{4\nu}\rho\sigma r_0 \operatorname{Re} B_{+} + \frac{1}{2}\rho\sigma r_0 \operatorname{Im}\int_0^T z r_{+} e^{-\nu z}\, dz \tag{3.7}$$

The second integral in (3.7) can, with the aid of (1.16), be represented thus:

$$\int_0^T z r_{+} e^{-\nu z} dz = AT^2 I_A + \frac{2}{\pi} BT^3 I_B + CT^4 I_C \tag{3.8}$$

where I_A, I_B and I_C denote the following

$$I_A = S_0(-\mu) - S_1(-\mu), \quad I_B = \frac{1}{\mu} + \left(1 + \frac{2}{\mu^2}\right)S_0(-\mu) - 2S_1(-\mu) + \frac{2}{\mu^3} S_0^{-1}(-\mu)$$

$$I_C = \frac{1}{\mu} + \left(\frac{1}{2} + \frac{3}{\mu^2}\right) S_0(-\mu) - 3\left(\frac{1}{2} + \frac{2}{\mu^4}\right) S_1(-\mu)$$

$$S_0(-\mu) = \int_0^1 \frac{e^{-\mu u} du}{\sqrt{1-u^2}} = \frac{\pi}{2}\left[I_0(\mu) - L_0(\mu)\right] \tag{3.9}$$

$$S_1(-\mu) = \int_0^1 \sqrt{1-u^2}\, e^{-\mu u} du = \frac{\pi}{2\mu}\left[I_1(\mu) - L_1(\mu)\right]$$

$$S_0^{-1}(-\mu) = \int_0^1 \frac{e^{-\mu u} - 1}{u\sqrt{1-u^2}} du = -\int_0^{\mu} S_0(-\mu)\, d\mu$$

It follows from formulas (1.8) and (1.11) that at point $x = T$ the velocity of the particles of liquid, in general, becomes infinite, and in the neighborhood of this point the velocity function is as follows:

$$\frac{dw}{dx} e^{j\sigma t} = i\sqrt{\frac{T}{2(x-T)}}\left(A + \frac{2}{\pi} BT + \frac{1}{2} CT^2\right) e^{j\sigma t} + F(x) e^{j\sigma t}$$

where the function $F(x)$ remains finite at $x = T$.

The presence of the infinite velocity, or in practice, very high velocity, demonstrates that in the neighborhood of the point $x = T$, there is a low pressure region, as a result of which the liquid will act as a concentrated suction force downward. The magnitude of this force is determined by the formula [2]:

$$Z = -\rho\pi\left[(x-T)\left(\frac{dw}{dx} e^{j\sigma t}\right)^2\right]_{x=T}$$

From this formula and from the rule in (3.3) we arrive at the following expression for the mean value of the suction force.

$$Z_{\lambda}^{*} = \frac{\pi}{4}\rho T\left|A + \frac{2}{\pi} BT + \frac{1}{2} CT^2\right|^2 \tag{3.10}$$

The expressions derived in this paper allow mean values of hydrodynamic forces to be calculated for practical cases. Mean values of the rise of the free surface of the liquid can also be evaluated in quadratic approximation from the general formulas quoted in [7] (also in [6]).

Similar calculations of the basic characteristics of waves and hydrodynamic forces for the case of pure diffraction can be found in [1].

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